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Integrable spin chains with $\mathrm{U}(1)^3$ symmetry and generalized Lunin-Maldacena backgrounds

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Abstract: We consider the most general three-state spin chain with $\mathrm{U}(1)^3$ symmetry and nearest neighbour interaction. Our model contains as a special case the spin chain describing the holomorphic three scalar sector of the three parameter complex deformation of $\mathcal{N} = 4$ SYM, dual to type IIB string theory in the generalized Lunin-Maldacena backgrounds discovered by Frolov. We formulate the coordinate space Bethe ansatz, calculate the S-matrix and determine for which choices of parameters the S-matrix fulfills the Yang-Baxter equations. For these choices of parameters we furthermore write down the R-matrix. We find in total four classes of integrable models. In particular, each already known model of the above type is nothing but one in a family of such models.

Keywords: Bethe Ansatz, Integrable Field Theories, AdS-CFT Correspondence.

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1. Introduction

Integrability has played a prominent role in recent years exploration of the AdS/CFT correspondence [1], the starting signal being the discovery by Minahan and Zarembo that the planar one-loop dilatation operator of the $\mathfrak{so}(6)$ sector of $\mathcal{N} = 4$ SYM could be identified with the hamiltonian of an integrable spin chain [2]. This discovery was soon extended beyond the $\mathfrak{so}(6)$ sector [3] and beyond the one-loop order [4]. On the string theory side integrable structures first made their appearance with the observation that the Green-Schwarz superstring on $AdS_5 \times S^5$ possessed an infinite set of non-local conserved charges [5] and were more precisely described in a series of important papers [6 − 11]. An interesting question is to what extent the existence of integrable structures in the gauge-string duality is linked to supersymmetry.[1] A first possibility to study this question was provided with the discovery by Lunin and Maldacena [13] of the background required to make type IIB string theory dual to the Leigh-Strassler β-deformation of $\mathcal{N} = 4$ SYM [14], a conformal quantum field theory carrying $\mathcal{N} = 1$ supersymmetry. Later the construction of Lunin and Maldacena was generalized to a three parameter complex deformation of $\mathcal{N} = 4$ SYM which is still conformal (at least to leading order in N) but for which supersymmetry is lost [15]. Furthermore, it was shown that for uni-modular deformation parameters one can construct a Lax pair for (the bosonic part of) the corresponding classical string theory, showing that

[1] It is of course also interesting to study the existence of integrable structures for gauge theories for which no string theory dual is currently known. For a review and further references, see [12].

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this theory like its undeformed counterpart is integrable [15]. (A Lax pair for the Landau-Lifshitz model describing fast rotating three-spin strings in the deformed background [16] was derived in reference [17]). Let us denote the three deformation parameters as $r_j e^{i\gamma_j}$, with $r_j > 0$, γ_j real, $j \in \{0,1,2\}$. (For the precise definition, see section 2.) The three parameter deformation of $\mathcal{N} = 4$ SYM reduces to the Leigh-Strassler one when the three deformation parameters are identical. In [18] the general deformation process was studied from the gauge theory point of view and it was found that the case $r_0 = r_1 = r_2 = 1$ always leads to an integrable dilatation operator (at least to the order that the latter is known). In addition, exact matching of gauge theory anomalous dimensions and string state energies was found in the two- as well as the three-scalar holomorphic sub-sectors (the analogues of the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ sub-sectors of the undeformed theory) [16]. Results on the existence of integrability on the gauge theory side for general deformation parameters are mostly negative. For instance, in the two scalar holomorphic sector integrability holds at one loop order for $r_0 = r_1 = r_2 = r$ and $\gamma_0 = \gamma_1 = \gamma_2 = \gamma$ but is very likely to break down already at two-loop order [19]. Furthermore, it was found in [20], see also [21], that assuming $r_0 = r_1 = r_2 = r$ integrability can only be obtained at one loop order in the three scalar holomorphic sector if $r = 1$. The one-loop dilatation operator in the three scalar holomorphic sector of the three parameter complex deformation of the gauge theory is a three-state spin chain with nearest neighbour interaction and $\mathrm{U}(1)^3$ symmetry. In the present paper we study the most general spin chain with these properties and determine for which choices of parameters it can be integrable. We find four classes of integrable models. In particular each already known integrable model of the above type is nothing but one in a family of such models. We also rule out the possibility that the complex three parameter deformation of the one-loop planar $\mathfrak{su}(3)$ sector of $\mathcal{N} = 4$ SYM could be integrable under more general circumstances than those already identified.

The organization of our paper is as follows. We begin in section 2 by writing down the most general hamiltonian for the type of model we wish to consider. Subsequently, in section 3 we determine its S-matrix and in section 4 investigate for which choices of parameters this S-matrix fulfills the Yang-Baxter relation. For these choices of parameters we then in section 5 construct an R-matrix for the model. Finally, we compare our integrable models to those already known in section 6. We finish with a brief conclusion.

2. The model

We consider the most general 3-state spin chain with nearest neighbour interaction and $\mathrm{U}(1)^3$ symmetry. We denote the basis of states at a given site as $|0\rangle$, $|1\rangle$, and $|2\rangle$. A $\mathrm{U}(1)^3$ symmetric hamiltonian conserves the number of states of each type and accordingly takes the form

$$\begin{aligned} H = {} & H_{00}^{00} E_{00} E_{00} + H_{11}^{11} E_{11} E_{11} + H_{22}^{22} E_{22} E_{22} + H_{12}^{12} E_{11} E_{22} + H_{21}^{12} E_{12} E_{21} + \\ & + H_{12}^{21} E_{21} E_{12} + H_{21}^{21} E_{22} E_{11} + H_{01}^{10} E_{10} E_{01} + H_{10}^{10} E_{11} E_{00} + H_{01}^{01} E_{00} E_{11} \\ & + H_{10}^{01} E_{01} E_{10} + H_{02}^{20} E_{20} E_{02} + H_{20}^{20} E_{22} E_{00} + H_{02}^{02} E_{00} E_{22} + H_{20}^{02} E_{02} E_{20} \,, \end{aligned} \tag{2.1}$$

where $E_{jk}E_{lm}$ is an abbreviation for $\sum_{i=1}^{L} E^{i}_{jk}E^{i+1}_{lm}$, with L being the length of the spin chain, and where $E_{ij}|k\rangle = |i\rangle\delta_{jk}$, i.e. E_{ij} are the generators of $\mathfrak{gl}(3)$. The spin chain is assumed to be closed. Restricting ourselves to hermitean hamiltonians we shall assume that the diagonal elements are real and that the off-diagonal ones can be written as

$$H^{12}_{21} = (H^{21}_{12})^* \equiv r_0 e^{i\gamma_0}\,, \qquad H^{20}_{02} = (H^{02}_{20})^* \equiv r_2 e^{i\gamma_2}\,, \qquad H^{01}_{10} = (H^{10}_{01})^* \equiv r_1 e^{i\gamma_1}\,, \tag{2.2}$$

where r_0, r_1 and r_2 are real and positive and where γ_0, γ_1 and γ_2 are real. Our aim is to determine for which values of the parameters this hamiltonian is integrable. An interesting spin chain which is included in the class of models given by eqn. (2.1) is the spin chain describing the the three scalar holomorphic sub-sector of the three parameter complex deformation of $\mathcal{N} = 4$ SYM, dual to type IIB string theory on the generalized Lunin-Maldacena backgrounds found by Frolov. For this model one has

$$\begin{aligned}
&H^{00}_{00} = H^{11}_{11} = H^{22}_{22} = 0\,, \qquad H^{12}_{12} = H^{01}_{01} = H^{20}_{20} = 1\,,\\
&H^{21}_{21} = r_0^2\,, \qquad H^{02}_{02} = r_1^2\,, \qquad H^{10}_{10} = r_2^2\,,\\
&H^{12}_{21} = (H^{21}_{12})^* \equiv r_0 e^{i\gamma_0}\,, \qquad H^{01}_{10} = (H^{10}_{01})^* \equiv r_1 e^{i\gamma_1}\,, \qquad H^{20}_{02} = (H^{02}_{20})^* \equiv r_2 e^{i\gamma_2}\,.
\end{aligned} \tag{2.3}$$

In [20] it was found that (2.3) is integrable for $\gamma_0 = \gamma_1 = \gamma_2 = \gamma = \frac{2n\pi}{L}$ with n integer and $r_0 = r_1 = r_2 = r$ only if $r = 1$. Later the model was shown to be integrable for $r_0 = r_1 = r_2 = 1$ for arbitrary values of γ_0, γ_1 and γ_2 [18]. The case $r_0 = r_1 = r_2 = 1$, $\gamma_0 = \gamma_1 = \gamma_2 = \pi$ is the usual XXX $\mathfrak{su}(3)$ spin chain. An additional number of integrable special cases of the model (2.1) are already known. We shall return to these in section 6. The problem of studying the most general spin chain with certain global conservation laws has also been pursued in condensed matter physics, see for instance [22]. Very recently, another problem of a somewhat similar nature was addressed, namely the problem of determining the most general integrable long range spin chain with the spins transforming in the fundamental of $\mathfrak{gl}(n)$ [23].

Integrability implies the existence of an R-matrix, $R(u)$, depending on a spectral parameter u and fulfilling

$$R(u)|_{u=u_0} = \mathcal{P}\,, \qquad \mathcal{P}\frac{d}{du}R(u)|_{u=u_0} = H\,, \tag{2.4}$$

where $\mathcal{P}$ is the permutation operator. Furthermore, the R-matrix must obey the Yang Baxter equation

$$R^{j_1j_2}_{i_1i_2}(u-v)R^{k_1j_3}_{j_1i_3}(u)R^{k_2k_3}_{j_2j_3}(v) = R^{j_2j_3}_{i_2i_3}(v)R^{j_1k_3}_{i_1j_3}(u)R^{k_1k_2}_{j_1j_2}(u-v)\,. \tag{2.5}$$

We can write a hamiltonian H of the form (2.1) with the angles γ_0, γ_1 and γ_2 present in terms of a hamiltonian without angles, $\tilde{H}$,

$$H^{kl}_{ij} = \exp\left(\frac{i}{2}(\epsilon_{ijm}\gamma_m - \epsilon_{kln}\gamma_n)\right)\tilde{H}^{kl}_{ij}\,. \tag{2.6}$$

Then, if the hamiltonian H admits an R-matrix, R^{kl}_{ij}, we can construct an R-matrix, $\tilde{R}^{kl}_{ij}$, corresponding to $\tilde{H}$ by the following recipe [24, 18]

$$\tilde{R}^{kl}_{ij} = \exp\left(\frac{i}{2}(\epsilon_{ijm}\gamma_m - \epsilon_{kln}\gamma_n)\right)R^{lk}_{ij}\,. \tag{2.7}$$

Conversely, given an R-matrix corresponding to a hamiltonian of the form (2.1) with the angles set to zero, one can construct an R-matrix for a hamiltonian with the angles re-introduced by applying the transformation inverse to (2.7). Thus any model integrable without the presence of angles is also integrable with arbitrary values of the angles and any model integrable with non-zero values of the angles is also integrable if the angles are set to zero. In other words, when searching for integrable models, one can leave out the angles from the analysis.

To test for integrability in a system that admits scattering one can, instead of directly constructing an R-matrix, study the properties of an appropriately defined S-matrix. In the present case the S-matrix can be written down by choosing a reference state, say, $|0\ldots 0\rangle$ and considering scattering of excitations of type 1 and 2. Integrability normally requires non-diffractive and factorized scattering.[2] This implies that the S-matrix must fulfill the Yang-Baxter like relation

$$S_{2,3}(p_2,p_3)S_{1,3}(p_1,p_3)S_{1,2}(p_1,p_2) = S_{1,2}(p_1,p_2)S_{1,3}(p_1,p_3)S_{2,3}(p_2,p_3)\,, \tag{2.8}$$

where each S-matrix acts in a eight-dimensional space. For a graphical representation of this relation, see the appendix. In the following we shall follow the simpler route of first studying the S-matrix. Hence we start by considering two particle scattering, write down the coordinate space Bethe ansatz, determine the S-matrix and investigate for which choices of parameters it fulfills the relation (2.8). Afterwards, we write down the corresponding R-matrix. The coordinate space Bethe ansatz was introduced by Bethe in 1931 [25] and revived by Staudacher in connection with the study of $\mathcal{N}=4$ SYM in [26]. In reference [22] a model similar to ours was analyzed using the so-called matrix product ansatz, but the analysis was not taken to the point of actually determining and classifying all integrable cases.

3. The Bethe ansatz and the S-matrix

It is obvious that the three states $|0\ldots 0\rangle$, $|1\ldots 1\rangle$ and $|2\ldots 2\rangle$ are all eigenstates of the hamiltonian. In the following we shall choose $|0\ldots 0\rangle$ as our reference state. The integrability properties of the model should not depend on the choice of reference state. First, let us study states containing only one excitation. We define

$$|1\rangle = \sum_{1\le l_1\le L} \psi_1(l_1)|00\overset{\overset{l_1}{\downarrow}}{1}000\rangle\,, \tag{3.1}$$

and similarly for $|2\rangle$. Writing down the Schrödinger equation gives

$$\left(H^{00}_{00}(L-2)+H^{10}_{10}+H^{01}_{01}\right)\psi_1(l_1)+r_1\left(\psi_1(l_1+1)+\psi_1(l_1-1)\right)=E_1\psi_1(l_1)\,. \tag{3.2}$$

This equation is immediately solved by the plane wave

$$\psi_1(l_1)=e^{ip_1l_1}\,,\qquad E_1=H^{00}_{00}(L-2)+H^{10}_{10}+H^{01}_{01}+r_1\left(e^{ip_1}+e^{-ip_1}\right). \tag{3.3}$$

[2] In the case where the angles are present in (2.2) the scattering is actually diffractive, see the appendix. Due to the argument given above, the model can nevertheless be integrable.

Similarly, we find a one excitation eigenstate $|2\rangle$ given by

$$\psi_2(l_2) = e^{ip_2 l_2}\,, \qquad E_2 = H^{00}_{00}(L-2) + H^{20}_{20} + H^{02}_{02} + r_2\left(e^{ip_2} + e^{-ip_2}\right). \tag{3.4}$$

We notice that in general the two types of excitations have different dispersion relations. Whereas a difference in the p-independent terms is harmless, a difference in the p-dependent terms is normally viewed as a signal that the model is not integrable. We shall make this statement more precise below.

Let us now turn to studying two-body interactions. We choose as a basis for the space of two-particle excitations the states $|11\rangle$, $|12\rangle$, $|21\rangle$ and $|22\rangle$ where $|ij\rangle$ is defined as

$$|ij\rangle = \sum_{1\le l_1<l_2\le L} \psi_{ij}(l_1,l_2)|00\ \overset{l_1}{\overset{\downarrow}{i}}\ 000\ \overset{l_2}{\overset{\downarrow}{j}}\ 000\rangle\,. \tag{3.5}$$

When $r_0 \neq 0$ the model allows for scattering between particles of type 1 and 2 and the states $|12\rangle$ and $|21\rangle$ will mix. Accordingly, our S-matrix will take the form

$$S = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & \bar{b} & \bar{c} & 0 \\ 0 & 0 & 0 & d \end{pmatrix}. \tag{3.6}$$

We shall first consider the case $r_0 \neq 0$. Later, we will comment on the simpler case $r_0 = 0$.

Let us to begin with study the states $|11\rangle$ and $|22\rangle$ which do not mix with anything. For the former one we get from the Schrödinger equation

$$\begin{aligned}
&l_2 > l_1+1: \\
&E_{11}\psi_{11}(l_1,l_2) = \left(H^{00}_{00}(L-4) + 2H^{10}_{10} + 2H^{01}_{01}\right)\psi_{11}(l_1,l_2) + \\
&\qquad + r_1\{\psi_{11}(l_1+1,l_2) + \psi_{11}(l_1,l_2+1) + \\
&\qquad\qquad + \psi_{11}(l_1-1,l_2) + \psi_{11}(l_1,l_2-1)\}\,, \qquad (3.7)\\
&l_2 = l_1+1: \\
&E_{11}\psi_{1}1(l_1,l_2) = \left(H^{00}_{00}(L-3) + H^{11}_{11} + H^{10}_{10} + H^{01}_{01}\right)\psi_{11}(l_1,l_2) + \\
&\qquad + r_1\left\{\psi_{11}(l_1,l_2+1) + \psi_{11}(l_1-1,l_2)\right\}. \qquad (3.8)
\end{aligned}$$

Inserting the standard Bethe ansatz

$$\psi_{11}(l_1,l_2) = e^{ip_1 l_1 + ip_2 l_2} + a(p_2,p_1)e^{ip_1 l_2 + ip_2 l_1}\,, \tag{3.9}$$

we find

$$E_{11} = H^{00}_{00}(L-4) + 2H^{10}_{10} + 2H^{01}_{01} + r_1\left(e^{ip_1} + e^{ip_2} + e^{-ip_1} + e^{-ip_2}\right), \tag{3.10}$$

and

$$a(p_1,p_2) = -\frac{\sigma_1 e^{ip_1} + r_1 e^{ip_1+ip_2} + r_1}{\sigma_1 e^{ip_2} + r_1 e^{ip_1+ip_2} + r_1}\,, \tag{3.11}$$

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where

$$\sigma_1 = H^{10}_{10} - H^{00}_{00} - H^{11}_{11} + H^{01}_{01} \,. \tag{3.12}$$

The results from 22-scattering can be obtained from those of 11-scattering by making the replacements $H^{10}_{10} \to H^{20}_{20}$, $H^{01}_{01} \to H^{02}_{02}$, $r_1 \to r_2$. Thus we use the Bethe ansatz

$$\psi_{22}(l_1, l_2) = e^{ip_1 l_1 + ip_2 l_2} + d(p_2, p_1) e^{ip_1 l_2 + ip_2 l_1} \,, \tag{3.13}$$

and get

$$E_{22} = H^{00}_{00}(L-4) + 2H^{20}_{20} + 2H^{02}_{02} + r_2 \left(e^{ip_1} + e^{ip_2} + e^{-ip_1} + e^{-ip_2} \right) , \tag{3.14}$$

and

$$d(p_1, p_2) = -\frac{\sigma_2 e^{ip_1} + r_2 e^{ip_1 + ip_2} + r_2}{\sigma_2 e^{ip_2} + r_2 e^{ip_1 + ip_2} + r_2} \,, \tag{3.15}$$

where

$$\sigma_2 = H^{20}_{20} - H^{00}_{00} - H^{22}_{22} + H^{02}_{02} \,. \tag{3.16}$$

Next, let us turn to studying the states which mix. For these the Schrödinger equation gives rise to the relations

$$\begin{aligned}
&l_2 > l_1 + 1 : \\
E\psi_{12}(l_1, l_2) \;&=\; \left(H^{00}_{00}(L-4) + H^{10}_{10} + H^{01}_{01} + H^{20}_{20} + H^{02}_{02} \right) \psi_{12}(l_1, l_2) + \\
&\quad + r_1 \left(\psi_{12}(l_1+1, l_2) + \psi_{12}(l_1-1, l_2) \right) + \\
&\quad + r_2 \left(\psi_{12}(l_1, l_2+1) + \psi_{12}(l_1, l_2-1) \right) , \qquad (3.17) \\
E\psi_{21}(l_1, l_2) \;&=\; \left(H^{00}_{00}(L-4) + H^{10}_{10} + H^{01}_{01} + H^{20}_{20} + H^{02}_{02} \right) \psi_{21}(l_1, l_2) + \\
&\quad + r_1 \left(\psi_{21}(l_1, l_2+1) + \psi_{21}(l_1, l_2-1) \right) + \\
&\quad + r_2 \left(\psi_{21}(l_1+1, l_2) + \psi_{21}(l_1-1, l_2) \right) , \qquad (3.18) \\
&l_2 = l_1 + 1 : \\
E\psi_{12}(l_1, l_2) \;&=\; \left(H^{00}_{00}(L-3) + H^{12}_{12} + H^{01}_{01} + H^{20}_{20} \right) \psi_{12}(l_1, l_2) \\
&\quad + r_0 \psi_{21}(l_1, l_2) + r_1 \psi_{12}(l_1-1, l_2) + r_2 \psi_{12}(l_1, l_2+1) \,, \qquad (3.19) \\
E\psi_{21}(l_1, l_2) \;&=\; \left(H^{00}_{00}(L-3) + H^{21}_{21} + H^{10}_{10} + H^{02}_{02} \right) \psi_{21}(l_1, l_2) + \\
&\quad + r_0 \psi_{12}(l_1, l_2) + r_1 \psi_{21}(l_1, l_2+1) + r_2 \psi_{21}(l_1-1, l_2) \,. \qquad (3.20)
\end{aligned}$$

Let us now choose the following Bethe ansatz

$$\begin{aligned}
\psi_{12} \;&=\; A_{12} e^{ip_1 l_1 + ip_2 l_2} + A'_{12} e^{ip'_1 l_2 + ip'_2 l_1} \,, \qquad (3.21) \\
\psi_{21} \;&=\; A_{21} e^{ip'_1 l_1 + ip'_2 l_2} + A'_{21} e^{ip_1 l_2 + ip_2 l_1} \,, \qquad (3.22)
\end{aligned}$$

where due to the translational invariance of our model

$$p_1 + p_2 = p'_1 + p'_2 \,. \tag{3.23}$$

The idea is that two excitations with momenta p_1 and p_2 can scatter whereby their momenta get changed to p'_1 and p'_2. Inserting this into (3.17) and (3.18) we find for the energy

$$\begin{aligned}
E \;&=\; H^{00}_{00}(L-4) + H^{10}_{10} + H^{01}_{01} + H^{20}_{20} + H^{02}_{02} + r_1 (e^{ip_1} + e^{-ip_1}) + r_2 (e^{ip_2} + e^{-ip_2}) \\
&=\; H^{00}_{00}(L-4) + H^{10}_{10} + H^{01}_{01} + H^{20}_{20} + H^{02}_{02} + r_1 (e^{ip'_2} + e^{-ip'_2}) + r_2 (e^{ip'_1} + e^{-ip'_1}) \,. \qquad (3.24)
\end{aligned}$$

Equations (3.24) and (3.23) determine p_1' and p_2'

$$e^{ip_1'} = e^{ip_1}\frac{r_2 + r_1 e^{ip_1+ip_2}}{r_1 + r_2 e^{ip_1+ip_2}}, \tag{3.25}$$

$$e^{ip_2'} = e^{ip_2}\frac{r_1 + r_2 e^{ip_1+ip_2}}{r_2 + r_1 e^{ip_1+ip_2}}, \tag{3.26}$$

where the solutions are chosen such as to reproduce the standard case $p_1' = p_1$, $p_2' = p_2$ when $r_1 = r_2 \neq 0$. If both r_1 and r_2 vanish the usual Bethe ansatz $p_1' = p_1$ and $p_2' = p_2$ is still applicable. If only one of the two vanishes one necessarily has $p_1' = p_2$ and $p_2' = p_1$, i.e. scattering between excitations of type 1 and 2 is not possible. In this case one should choose another reference state. However, the results for this situation can be obtained by symmetry arguments from those of the case $r_0 = 0$ which we will consider below.

The S-matrix elements involved in the 12-scattering are defined, using the transmission diagonal representation, as

$$\begin{pmatrix} A_{21}' \\ A_{12}' \end{pmatrix} = \begin{pmatrix} c(p_2,p_1) & b(p_2,p_1) \\ \bar{b}(p_2,p_1) & \bar{c}(p_2,p_1) \end{pmatrix} \begin{pmatrix} A_{12} \\ A_{21} \end{pmatrix}, \tag{3.27}$$

and can be found from (3.19) and (3.20) which with the Bethe ansatz (3.21) and (3.22) read

$$\begin{aligned} 0 = {} & A_{21} r_0 e^{ip_2'} + A_{21}' r_0 e^{ip_1} - \\ & - A_{12}\left\{\tau_1 + r_1 e^{ip_1} + r_2 e^{-ip_2}\right\} e^{ip_2} - A_{12}'\left\{\tau_1 + r_1 e^{ip_2'} + r_2 e^{-ip_1'}\right\} e^{ip_1'}, \end{aligned} \tag{3.28}$$

$$\begin{aligned} 0 = {} & A_{12} r_0 e^{ip_2} + A_{12}' r_0 e^{ip_1'} - \\ & - A_{21}\left\{\tau_2 + r_1 e^{-ip_2'} + r_2 e^{ip_1'}\right\} e^{ip_2'} - A_{21}'\left\{\tau_2 + r_1 e^{-ip_1} + r_2 e^{ip_2}\right\} e^{ip_1}, \end{aligned} \tag{3.29}$$

where

$$\tau_1 = H_{10}^{10} - H_{00}^{00} - H_{12}^{12} + H_{02}^{02}, \tag{3.30}$$

$$\tau_2 = H_{01}^{01} - H_{00}^{00} - H_{21}^{21} + H_{20}^{20}. \tag{3.31}$$

As mentioned above it is common lore that the model can not be integrable unless the two scattering excitations have the same dispersion relation. We will give the precise argument for our hamiltonian in the appendix.[3] From now on we assume that $r_1 = r_2$. If in addition $r_0 \neq 0$ scattering will also be possible with the choice of either of the states $|1\ldots1\rangle$ or $|2\ldots2\rangle$ as reference state and it follows by symmetry arguments that $r_0 = r_1 = r_2 = r$. For $r_0 = r_1 = r_2 = r \neq 0$ the remaining S-matrix elements read

$$c(p_1,p_2) = \bar{c}(p_1,p_2) = \frac{1}{D}\left(e^{ip_1} - e^{ip_2}\right)\left(1 + e^{ip_1+ip_2}\right), \tag{3.32}$$

$$\begin{aligned} b(p_1,p_2) = {} & -\frac{1}{D}((t_1 t_2 - 1)e^{ip_1+ip_2} + (1 + e^{ip_1+ip_2})^2 + \\ & + \left(t_1 e^{ip_2} + t_2 e^{ip_1}\right)(1 + e^{ip_1+ip_2})), \end{aligned} \tag{3.33}$$

$$\begin{aligned} \bar{b}(p_1,p_2) = {} & -\frac{1}{D}((t_1 t_2 - 1)e^{ip_1+ip_2} + (1 + e^{ip_1+ip_2})^2 + \\ & + \left(t_2 e^{ip_2} + t_1 e^{ip_1}\right)(1 + e^{ip_1+ip_2})), \end{aligned} \tag{3.34}$$

[3] Integrable hamiltonians of the type (2.1) with generic values of the angles, however, constitute an exception to the rule, see the appendix.

where

$$D(p_1,p_2) = \left((t_1t_2-1)e^{2ip_2} + (1+e^{ip_1+ip_2})^2 + (t_1+t_2)e^{ip_2}(1+e^{ip_1+ip_2})\right), \tag{3.35}$$

and

$$t_1 = \frac{\tau_1}{r}, \qquad t_2 = \frac{\tau_2}{r}. \tag{3.36}$$

It is easy to verify that the S-matrix given by the relations (3.11), (3.15), (3.32), (3.33) and (3.34) is unitary. We notice that effectively the S-matrix (with the restriction $r_0 = r_1 = r_2 = r \neq 0$) depends only on the four parameters

$$s_1 = \frac{\sigma_1}{r}, \qquad s_2 = \frac{\sigma_2}{r}, \qquad t_1, \qquad t_2, \tag{3.37}$$

(cf. eqs. (3.11), (3.15), (3.32), (3.33), (3.34) and (3.35)). This is easy to explain. To begin with we had 15 parameters, we then removed the angles as integrability properties can be analysed without them and set $r_0 = r_1 = r_2 = r$ since the model otherwise is necessarily non-integrable. This leaves us with 10 parameters, the single off-diagonal one, r, and 9 diagonal ones. First, one can of course make a global rescaling by $1/r$ without changing the scattering properties of the model. Secondly, one can construct the following number operators

$$\hat{N}_0 = 1 \otimes E_{00}, \qquad \hat{M}_0 = E_{00} \otimes 1, \tag{3.38}$$

$$\hat{N}_1 = 1 \otimes E_{11}, \qquad \hat{M}_1 = E_{11} \otimes 1, \tag{3.39}$$

$$\hat{N}_2 = 1 \otimes E_{22}, \qquad \hat{M}_2 = E_{22} \otimes 1, \tag{3.40}$$

where $\hat{N}_i$ and $\hat{M}_i$ counts the number of particles of type i. Only five of these operators are independent since we have the relation

$$\hat{N}_0 + \hat{N}_1 + \hat{N}_2 = \hat{M}_0 + \hat{M}_1 + \hat{M}_2. \tag{3.41}$$

Since the number operators commute with the hamiltonian, adding such operators will not change the scattering properties of the system. This means that effectively the S-matrix depends only on four parameters.

When $r_1 = r_2 = 0$ we have

$$c(p_1,p_2) = \bar{c}(p_1,p_2) = 0 \qquad b(p_1,p_2) = \bar{b}(p_1,p_2) = a(p_1,p_2) = d(p_1,p_2) = -e^{ip_1-ip_2}. \tag{3.42}$$

Furthermore, for $r_0 = 0$, $r_1 = r_2 = r$ the S-matrix elements involved in 12-scattering read

$$c(p_1,p_2) = \bar{c}(p_1,p_2) = 0, \tag{3.43}$$

$$b(p_1,p_2) = -\frac{t_2e^{ip_1} + e^{ip_1+ip_2} + 1}{t_2e^{ip_2} + e^{ip_2+ip_1} + 1}, \tag{3.44}$$

$$\bar{b}(p_1,p_2) = -\frac{t_1e^{ip_1} + e^{ip_1+ip_2} + 1}{t_1e^{ip_2} + e^{ip_2+ip_1} + 1}. \tag{3.45}$$

4. Solution of the YBE's

A necessary condition for integrability is that the unitary S-matrix fulfills the Yang-Baxter equation

$$S_{2,3}(p_2,p_3)S_{1,3}(p_1,p_3)S_{1,2}(p_1,p_2) = S_{1,2}(p_1,p_2)S_{1,3}(p_1,p_3)S_{2,3}(p_2,p_3)\,, \tag{4.1}$$

where each S-matrix acts in a 8-dimensional Hilbert space.

For a S-matrix on the form (3.6) the Yang-Baxter equation gives rise to 7 relations between matrix elements,

$$\bar{b}b'\bar{b}'' - b\bar{b}'b'' = 0\,, \tag{4.2}$$
$$a\bar{b}'a'' - \bar{b}a'\bar{b}'' - \bar{c}\bar{b}'c'' = 0\,, \tag{4.3}$$
$$ca'b'' - ac'b'' + \bar{b}b'c'' = 0\,, \tag{4.4}$$
$$\bar{b}\bar{c}'a'' - \bar{c}\bar{b}'b'' - \bar{b}a'\bar{c}'' = 0\,, \tag{4.5}$$
$$d\bar{b}'d'' - \bar{b}d'\bar{b}'' - c\bar{b}'\bar{c}'' = 0\,, \tag{4.6}$$
$$\bar{c}d'\bar{b}'' - d\bar{c}'\bar{b}'' + b\bar{b}'\bar{c}'' = 0\,, \tag{4.7}$$
$$\bar{b}c'd'' - c\bar{b}'b'' - \bar{b}d'c'' = 0\,, \tag{4.8}$$

where $b = b(p_1,p_2)$, $b' = b(p_1,p_3)$, $b'' = b(p_2,p_3)$, etc. At first sight these equations seem rather involved but a systematic investigation with the aid of Mathematica allows one to find the most general solution by purely *analytical* means. The simplest equation is equation (4.2) so this is the one to address first. Next, we note that for our case, where $c = \bar{c}$, equations (4.6)–(4.8) appear from equations (4.3)–(4.5) by the interchangement $a \to d$. Hence the solution of the former three equations can immediately be read off from the solution of the latter three.

The case $r_0 = r_1 = r_2 = r \neq 0$. In this situation the equation (4.2) gives that $t_2t_1 = 1$ and the equations (4.3), (4.4) and (4.5) give $s_1 = 0$ or $s_1 = t_1 + t_2$. Thus we find the following four families of integrable models

1. $t_2t_1 = 1$, $s_1 = s_2 = 0$.
2. $t_2t_1 = 1$, $s_1 = 0$, $s_2 = t_1 + t_2$.
3. $t_2t_1 = 1$, $s_1 = t_1 + t_2$, $s_2 = 0$.
4. $t_2t_1 = 1$, $s_1 = s_2 = t_1 + t_2$.

It is straightforward to check that these criteria for integrability do not depend on the choice of reference state. More precisely, the first three cases are related by Z_3 symmetry, i.e. they appear from one another when the labels 0, 1 and 2 are interchanged. The last is invariant under this symmetry. Thus, we have only two genuinely different classes of integrable models with $r_1 = r_2 = r_3 \neq 0$.

The case $r_0 \neq 0$, $r_1 = r_2 = 0$. In this case it is obvious that the YBE's are always fulfilled since all the S-matrix elements are identical.

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The case $r_0 = 0$, $r_1 = r_2 = r \neq 0$. Here the condition for integrability reads

$$t_1 = t_2 = s_1 = s_2 \,.$$

Finally, the model is obviously integrable for $r_0 = r_1 = r_2 = 0$.

5. R-matrices

In this section we write down the R-matrices corresponding to our integrable hamiltonians with $r_1 = r_2 = r_3 \neq 0$. These hamiltonians all have their off-diagonal elements equal to one. They can therefore be characterized entirely in terms of their diagonal elements. We can choose representatives for the four different cases above for instance as follows

1. $H_1 : (0, \frac{1}{t}, \frac{1}{t}, t, t + \frac{1}{t}, \frac{1}{t}, t, t, t + \frac{1}{t})$,
2. $H_2 : (t + \frac{1}{t}, \frac{1}{t}, \frac{1}{t}, t, 0, \frac{1}{t}, t, t, t + \frac{1}{t})$,
3. $H_3 : (t + \frac{1}{t}, \frac{1}{t}, \frac{1}{t}, t, t + \frac{1}{t}, \frac{1}{t}, t, t, 0)$,
4. $H_4 : (t + \frac{1}{t}, \frac{1}{t}, \frac{1}{t}, t, t + \frac{1}{t}, \frac{1}{t}, t, t, t + \frac{1}{t})$,

where the lists are lists of diagonal elements. Any member of a given class can be brought on the form above by addition of appropriate linear combinations of number operators. We can write the R-matrix for the models above in a collective form as

$$\begin{aligned}
R^{00}_{00}(u,v) &= A(u,v,s')\,, \qquad R^{11}_{11}(u,v) = A(u,v,\tilde{s})\,, \qquad R^{22}_{22}(u,v) = A(u,v,s)\,, \\
R^{12}_{12}(u,v) &= R^{21}_{21}(u,v) = R^{02}_{02}(u,v) = R^{20}_{20}(u,v) = R^{01}_{01}(u,v) = R^{10}_{10}(u,v) = C(u,v)\,, \\
R^{10}_{01}(u,v) &= R^{20}_{02}(u,v) = R^{21}_{12}(u,v) = B(u,v,t)\,, \\
R^{01}_{10}(u,v) &= R^{02}_{20}(u,v) = R^{12}_{21}(u,v) = B\left(u,v,\frac{1}{t}\right),
\end{aligned}$$

where

$$A(u,v,s) = \frac{su + uv + 1}{sv + uv + 1}\,, \tag{5.1}$$

$$B(u,v,t) = \frac{1 + uv + \frac{1}{t}u + tv}{1 + uv + (\frac{1}{t} + t)v}\,, \tag{5.2}$$

$$C(u,v) = \frac{u - v}{1 + uv + (t + \frac{1}{t})v}\,, \tag{5.3}$$

and where the remaining R-matrix elements vanish. The appropriate choices of s, $\tilde{s}$ and s' for the four hamiltonians above are

1. $H_1 : \quad (s, s', \tilde{s}) = (t + \frac{1}{t}, 0, t + \frac{1}{t})$,
2. $H_2 : \quad (s, s', \tilde{s}) = (t + \frac{1}{t}, t + \frac{1}{t}, 0)$,

3. $H_3: \quad (s, s', \tilde{s}) = (0, t+\frac{1}{t}, t+\frac{1}{t})$,

4. $H_4: \quad (s, s', \tilde{s}) = (t+\frac{1}{t}, t+\frac{1}{t}, t+\frac{1}{t})$.

The R-matrix fulfills the necessary requirements, namely

$$R(u,u) = P\,, \tag{5.4}$$

where P is the permutation operator and

$$\left(1+v^2+\left(t+\frac{1}{t}\right)v\right)^{-1} H = P\partial_u R(u,v)|_{u=v}\,. \tag{5.5}$$

Finally, it also satisfies the Yang Baxter relation

$$R^{j_1j_2}_{i_1i_2}(u,v)R^{k_1j_3}_{j_1i_3}(u,w)R^{k_2k_3}_{j_2j_3}(v,w) = R^{j_2j_3}_{i_2i_3}(v,w)R^{j_1k_3}_{i_1j_3}(u,w)R^{k_1k_2}_{j_1j_2}(u,v)\,. \tag{5.6}$$

As indicated by the notation above the R-matrix has been constructed from pieces from the earlier determined S-matrix, cf. eqns. (3.6), (3.15), (3.11), (3.32), (3.33) and (3.34). In many situations one can perform a transformation from (u,v) to a new set of variables (λ,ν) such that $R(\lambda,\nu) = R(\lambda-\nu)$. For instance, for the $\mathfrak{su}(3)$ XXX spin chain, which belongs to case four and has $t=1$, this transformation reads

$$u = -\frac{\lambda-i}{\lambda+i}, \qquad v = -\frac{\nu-i}{\nu+i}\,. \tag{5.7}$$

6. Comparison to known models

Here we list a number of already known integrable models whose hamiltonian can be written on the form (2.1).

The first example of an integrable model of the form (2.1) which comes to mind is the XXX $\mathfrak{su}(3)$ spin chain. It is characterized by the elements of the hamiltonian taking the values $H^{00}_{00} = H^{11}_{11} = H^{22}_{22} = 0$, $H^{12}_{12} = H^{21}_{21} = H^{10}_{10} = H^{01}_{01} = H^{20}_{20} = H^{02}_{02} = 1$, $H^{12}_{21} = H^{21}_{12} = H^{10}_{01} = H^{01}_{10} = H^{20}_{02} = H^{02}_{20} = -1$ or equivalently $r=1$, $s_1 = s_2 = 2$, $t_1 = t_2 = 1$. (Note that in the latter notation the angles are removed from the analysis.) It thus belongs to family number four.

In the same family we find the integrable deformations of this spin chain, describing the dilatation operator of the three scalar holomorphic sub-sector of the three parameter complex deformation of $\mathcal{N}=4$ SYM, given by (2.3) with $r_0 = r_1 = r_2 = 1$ [20, 18]. They all have $s_1 = s_2 = 2$, $t_1 = t_2 = 1$.

Within our formalism we can of course investigate whether it is possible to achieve integrability for a more general class of complex deformations of $\mathcal{N}=4$ SYM. The most interesting case is the case where all three r-variables are non-vanishing. First, we have seen that integrability demands that $r_0 = r_1 = r_2 = r$. In reference [20] it was argued that if $r_0 = r_1 = r_2 = r$ one furthermore needs that $r=1$. This also follows from the results

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above. Namely, for the deformed model with $r_0 = r_1 = r_2 = r$ we have

$$s_1 = s_2 = \frac{1+r^2}{r}, \tag{6.1}$$

$$t_1 = \frac{2r^2-1}{r}, \tag{6.2}$$

$$t_2 = \frac{2-r^2}{r}, \tag{6.3}$$

and we immediately see that according to the conditions for integrability presented above the model can only be integrable if $r = 1$ since only in this case $t_1 t_2 = 1$.

In family four we also find the integrable $\mathfrak{su}_\mathfrak{q}(3)$ spin chain likewise studied in [20] for its possible connection to other deformations of $\mathcal{N} = 4$ SYM. This model is characterized by $H^{00}_{00} = H^{11}_{11} = H^{22}_{22} = 0$, $H^{12}_{12} = H^{01}_{01} = H^{02}_{02} = 1$, $H^{12}_{21} = H^{21}_{12} = H^{10}_{01} = H^{01}_{10} = H^{20}_{02} = H^{02}_{20} = r$, $H^{21}_{21} = H^{10}_{10} = H^{20}_{20} = r^2$ or equivalently $s_1 = s_2 = \frac{1+r^2}{r}$, $t_1 = r$, $t_2 = 1/r$.

In family three we find f. inst. the $\mathfrak{su}(1|2)$ spin chain describing the $\mathfrak{su}(1|2)$ sub-sector of undeformed $\mathcal{N} = 4$ SYM and studied in [27]. Here one has $H^{00}_{00} = 0, H^{11}_{11} = 2$, $H^{22}_{22} = 0$, $H^{12}_{12} = H^{12}_{21} = H^{21}_{12} = H^{21}_{21} = H^{10}_{10} = H^{01}_{01} = H^{20}_{20} = H^{02}_{02} = 1$, $H^{10}_{01} = H^{01}_{10} = H^{20}_{02} = H^{02}_{20} = -1$ or equivalently $r = 1$, $s_1 = 0$, $s_2 = 2$, $t_1 = t_2 = 1$. The spin chain describing the analogue of the $\mathfrak{su}(1|2)$ sub-sector in the three parameter complex deformation of $\mathcal{N} = 4$ SYM theory is of course also included here. It has $s_1 = r + \frac{1}{r} - 2$, $s_2 = r + \frac{1}{r}$, $t_1 = 2r - \frac{1}{r}$, $t_2 = \frac{2}{r} - r$ and is only integrable if $r = 1$ where it reduces to the model considered in [18].

We finally note some models known from studies of integrability in the context of condensed matter systems, see for instance [22]. Choosing the parameters as $H^{00}_{00} = \epsilon_1 \cosh\gamma$, $H^{11}_{11} = \epsilon_2 \cosh\gamma$, $H^{22}_{22} = \epsilon_3 \cosh\gamma$, $H^{21}_{21} = H^{10}_{10} = H^{20}_{20} = -H^{12}_{12} = -H^{01}_{01} = -H^{02}_{02} = \sinh\gamma$ and $H^{12}_{21} = H^{21}_{12} = H^{10}_{01} = H^{01}_{10} = H^{20}_{02} = H^{02}_{20} = 1$. Choosing $\epsilon_1 = \epsilon_2 = \epsilon_3 = \pm 1$ this is the anisotropic Perk-Schultz model [28] which then belongs to class 4. Setting $\gamma = 0$ in this model we find the $\mathfrak{su}(3)$ Sutherland model [29]. Another example is the case when $\epsilon_1 = -\epsilon_2 = \epsilon_3 = 1$ which is the anisotropic supersymmetric t-J model [30], this model belongs to our class 2.

7. Conclusion

We have studied the most general three-state spin chain with nearest neighbour interaction and $\mathrm{U}(1)^3$ symmetry. This model contains as a special case the spin chain describing the three scalar holomorphic sub-sector of the three parameter complex deformation of $\mathcal{N} = 4$ SYM, dual to type IIB string theory in the generalized Lunin-Maldacena backgrounds found by Frolov [15]. We have made use of the conceptually simple coordinate space Bethe ansatz invented by Bethe in 1931 [25] and revived in connection with the study of $\mathcal{N} = 4$ SYM by Staudacher [26]. Solving the Yang Baxter relation for the S-matrix we identified four classes of integrable models. Subsequently we wrote down an R-matrix for the most interesting of these. Our findings show that each already known integrable model of the above type is nothing but one in a family of such models. We furthermore rule out the possibility that the complex three parameter deformation of the one-loop planar $\mathfrak{su}(3)$

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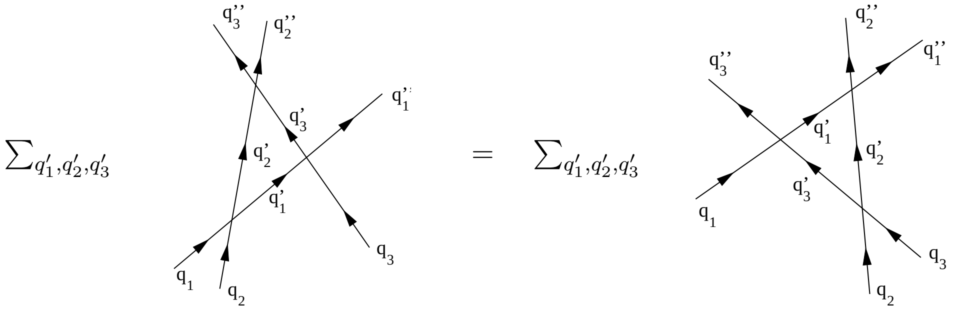

Figure 1: Graphical representation of the Yang-Baxter relations

sector of $\mathcal{N} = 4$ could be integrable under more general circumstances than those already identified.

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A. The case of different dispersion relations

The Yang Baxter relation (2.8) which describes the scattering between three excitatons has the well-known graphical interpretation shown in figure 1. A necessary condition for the relation to be fulfilled is that incoming and outgoing momenta are the same on the two sides of the equality sign. It is easy to find an example where this is not the case if $r_2 \neq r_1$. For $r_0 \neq 0$ one can f. inst. consider the situation shown in figure 2. Here the incoming particles are supposed to be of type 1, 1 and 2 with momenta q_1, q_2 and q_3. The outgoing particles are assumed to be of type 2, 1 and 1 and their momenta can be found by the use of (3.25). It is easy to see that the outgoing momenta on the two sides of the equation do not match unless $r_2 = r_1$. Since the integrability properties of the model can not depend on the choice of reference state the model can thus only be integrable if $r_0 = r_1 = r_2$. For $r_0 = 0$ excitations of type 1 and 2 can not cross each other but we can now instead consider the diagram shown in figure 3. Again, we reach the conclusion that $r_2 = r_1$.

Let us finally comment on the case where the angles are present. In that case the analogue of (3.25) reads:

$$e^{ip_1'} = e^{ip_1} e^{-i(\gamma_1+\gamma_2)} \frac{r_2 + r_1 e^{i(\gamma_2-\gamma_1)} e^{ip_1+ip_2}}{r_2 + r_1 e^{2i(\gamma_2-\gamma_1)} e^{ip_1+ip_2}}, \tag{A.1}$$

$$e^{ip_2'} = e^{ip_2} e^{i(\gamma_2+\gamma_1)} \frac{r_1 + r_2 e^{i(\gamma_2-\gamma_1)} e^{ip_1+ip_2}}{r_1 + r_2 e^{2i(\gamma_2-\gamma_1)} e^{ip_1+ip_2}}. \tag{A.2}$$

Setting $r_2 = r_1$ we find

$$p_1' = p_1 - (\gamma_2 + \gamma_1), \qquad p_2' = p_2 + (\gamma_2 + \gamma_1)\,. \tag{A.3}$$

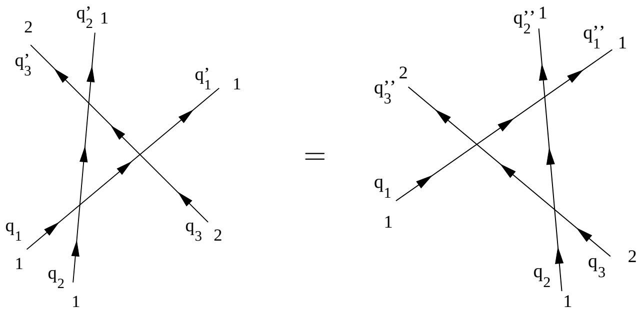

Figure 2: Factorized scattering requires this relation to hold. This is not possible unless the incoming and outgoing momenta are the same on both sides of the equality sign. Requiring that $q_i' = q_i''$ implies that $r_2 = r_1$.

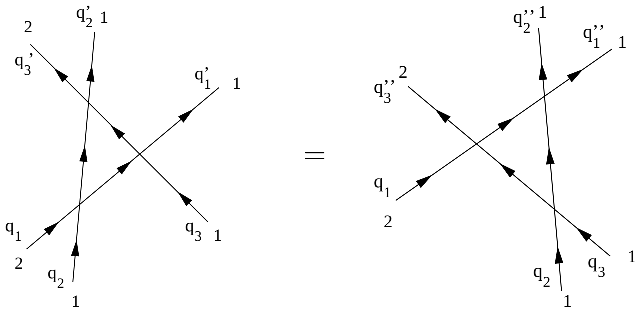

Figure 3: In the case $r_0 = 0$ this process determines that $r_1 = r_2$.

This means that the outgoing momenta in for instance figure 2 are

$$q_1' = q_1'' = q_1 - (\gamma_2 + \gamma_1)\,, \qquad q_2' = q_2'' = q_2 - (\gamma_2 + \gamma_1)\,, \qquad q_3' = q_3'' = q_3 + 2(\gamma_2 + \gamma_1)\,. \quad \text{(A.4)}$$

The outgoing momenta are hence the same on the two sides of the equation but the scattering is diffractive. However, if the system is integrable with the angles set to zero it is still integrable when the angles are introduced, cf. discussion on page 4.

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